

Covariant Renormalizable Anisotropic Theories and Off-Diagonal Einstein-Yang-Mills-Higgs Solutions

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Abstract – We use an important decoupling property of gravitational field equations in the general relativity theory and modifications, written with respect to nonholonomic frames with 2+2 spacetime decomposition. This allows us to integrate the Einstein equations (eqs) in very general forms with generic off-diagonal metrics depending on all spacetime coordinates via generating and integration functions containing (un-)broken symmetry parameters. We associate families of off-diagonal Einstein manifolds to certain classes of covariant gravity theories which have a nice ultraviolet behavior and seem to be (super) renormalizable in a sense of covariant modifications of Hořava-Lifshits gravity. The apparent breaking of Lorentz invariance is present in some "partner" anisotropically induced theories due to nonlinear coupling with effective parametric interactions determined by nonholonomic constraints and generic off-diagonal gravitational and matter fields configurations. Finally, we show how the constructions can be extended to include exact solutions for conjectured covariant renormalizable models with Einstein-Yang-Mills-Higgs fields.

Introduction. – One of the main difficulties in elaborating viable models of quantum gravity (QG) is that perturbations in the general relativity theory (GR) from a flat Minkowski spacetime result in non-renormalizable divergences from the ultraviolet region in momentum space. This problem is caused by the dimension of gravitational (Newton) constant and seems impossible to be solved for a unitary theory with four fundamental interactions which has Lorentz-invariance even some higher-derivative, spin-foam, loop type etc models of gravity may be renormalizable, see [1–3] for reviews and references. A recent new approach to QG is based on idea [4] to construct Lorentz non-invariant theories with scaling properties of space, x^i , and time, t , coordinates considered in the form $(\mathbf{x}, t) \rightarrow (b\mathbf{x}, b^z t)$, where $z = 2, 3, \dots$ In such cases, the ultraviolet (UV) behavior of the graviton propagator changes as $1/|\mathbf{k}|^2 \rightarrow 1/|\mathbf{k}|^{2z}$, where \mathbf{k} is the spacial momenta. It is possible to elaborate models of QG which are UV renormalizable, for instance, for $z = 3$ but with the price of introducing some terms breaking the Lorentz invariance explicitly. Due to the lack of full diffeomorphism invariance and, for instance, impossibility to exclude completely certain unphysical modes, such theories were criticized in a series of works (see, for instance, [5, 6]).

In [7], a covariant renormalizable gravity model developing the Hořava-like gravity to full diffeomorphism invariance was elaborated. In such an approach, the Lorentz-invariance of the graviton propagator is broken via a non-standard coupling with an unknown fluid. Some versions of such theories seem to be (super-) renormalizable, when physical transverse modes may appear. Certain applications in modern cosmology, with accelerating solutions and possible power law inflationary stage, seem to be of substantial interest.

The results of research in [8, 9] conclude that the gravitational field eqs in GR and various modifications can be decoupled with respect to certain classes of nonholonomic frames. It is possible to integrate the system of Einstein-Yang-Mills and Higgs (EYM-H) eqs in very general forms. For such solutions, the generic off-diagonal metrics can not be diagonalized via coordinate transforms and depend on all spacetime coordinates via corresponding generating and integration functions and possible (broken, or preserving) symmetry parameters. Choosing necessary types of such functions and parameters, via corresponding deformations of frame, metric and connection structures, and imposing non-integrable (nonholonomic) transforms, we can model effective nonlinear interactions, with scaling

properties and local anisotropies, which can be renormalizable.

We start our constructions with a set of actions $^{[i]}S = \frac{1}{\kappa^2} \int d^4u \sqrt{|\mathbf{g}|} \ ^{[i]}\mathcal{L}$ on a four dimensional (4-d) pseudo-Riemannian manifold V for Lagrangian

$$\begin{aligned} ^{[1]}\mathcal{L} &= {}^s\widehat{R} - \widehat{\Lambda} = ^{[2]}\mathcal{L} = {}^s\widetilde{R} + \widetilde{L} = ^{[3]}\mathcal{L} \\ &= R + L(T^{\mu\nu}, R_{\mu\nu}) = ^{[4]}\mathcal{L} = F(\check{R}). \end{aligned} \quad (1)$$

In our works, we use left up and low labels to geometric/physical objects which may have the same coefficients in a system of reference but subjected to different types of eqs for certain analogous/effective theories. All terms in such formulas are stated by the same metric structure $\mathbf{g} = \{g_{\alpha\beta}\}$ for standard and/or modified models of gravity theory generated by different (effective) Lagrangians when curvature scalars (${}^s\widehat{R}$, ${}^s\widetilde{R}$, R , \check{R}), cosmological constant $\widehat{\Lambda}$, non-standard coupling of Ricci, $R_{\mu\nu}$, and energy-momentum, $T^{\mu\nu}$, tensors via $L(\dots)$ and $F(\dots)$ etc. The above values and other possible constants and terms are such way postulated that $^{[1]}\mathcal{L}$ is used for constructing general classes of generic off-diagonal solutions in GR [8, 9] but $^{[3]}\mathcal{L}$ is of necessary type to get a covariant renormalization gravity as in [7]. For corresponding conditions on F the value $^{[4]}\mathcal{L}$ is related to modified theories [10]. To study QG models, the spacetime metric for theories derived from $^{[3]}\mathcal{L}$ and/or $^{[4]}\mathcal{L}$ will be taken, for simplicity, of type ${}^\diamond\mathbf{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ for perturbative models with flat background $\eta_{\alpha\beta}$. Such solutions are with breaking of Lorentz symmetry and, for well stated conditions, result into effective models with covariant renormalization. Using frame transforms $\mathbf{g}_{\alpha\beta} = e_{\alpha'}^{\alpha} e_{\beta'}^{\beta} {}^\diamond\mathbf{g}_{\alpha'\beta'}$,¹ we shall search for classical and quantum perturbative solutions for the theories determined by $^{[1]}\mathcal{L}$, or $^{[2]}\mathcal{L}$.

In this paper, we show how to construct off-diagonal solutions in GR (in general, with nontrivial effective anisotropic polarizations of cosmological constants), when in a theory for $^{[1]}S$ there are modelled effects with broken Lorenz invariance, non-standard effective anisotropic fluid coupling and behavior of the polarized propagator in the ultraviolet/infrared region. Such effects are derived to be similar to those for some (super-) renormalizable theories $^{[3]}S$ and/or $^{[4]}S$. The solutions for $^{[2]}S$ will be used as "bridges" between generating functions determining certain classes of generic off-diagonal Einstein manifolds and effective models with anisotropies and resulting violations for Lorentz symmetry. We state the conditions when the Einstein eqs transform nonlinearly, by imposing corresponding classes of nonholonomic constraints, to necessary types of effective systems of partial differential eqs (PDE) with parametric dependence of solutions which under quantization "survive" and stabilize in some

¹we label local coordinates and $u^\alpha = (x^i, y^a) \rightarrow u^{\alpha'} = (x^{i'}(u^\alpha), y^{a'}(u^\alpha))$, for indices $i, j, k, \dots = 1, 2$ and $a, b, c, \dots = 3, 4$ (in brief, we write $u = (x, y)$, $u' = (x', y')$); the Einstein rule on summation on repeating "up-low" indices will be applied if a contrary statement will be not emphasized

rescaled/anisotropic and renormalized forms. Finally, we shall generalize the constructions for exact and approximate anisotropic solutions for generic off-diagonal EYM systems and their covariant renormalizable models.

Decoupling property and formal integration of Einstein eqs. — We briefly review the anholonomic deformation method of constructing off-diagonal exact solutions in the GR theory and modifications [?, 9]. Any spacetime metric can be parametrized in a form

$$\begin{aligned} \mathbf{g} &= \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta = g_\alpha(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\alpha \\ &= g_{ij} e^i \otimes e^j + g_{ab} \mathbf{e}^a \otimes \mathbf{e}^b, \end{aligned} \quad (2)$$

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}, \quad (3)$$

$$\mathbf{e}_\alpha = [\mathbf{e}_i = \partial/\partial x^i - N_i^b(u) \partial_b, \mathbf{e}_a = \partial_a = \partial/\partial y^a], \quad (4)$$

$$\mathbf{e}^\beta = [e^j = dx^j, \mathbf{e}^b = dy^b + N_i^b(u) dx^i]. \quad (5)$$

We associate the coefficients $\mathbf{N} = \{N_i^a(u)\}$ to a 2+2 splitting of V stated explicitly for the tangent space $\mathbf{N}: TV = hV \oplus vV$ for a non-integrable (nonholonomic) distribution with conventional horizontal, h , and vertical, v , subspaces.² The decoupling property for the Einstein eqs and their solutions can be proved for any metric \mathbf{g} (2) and parametrization with "non-underlined" (in general, depending on variables (x^i, y^3)) and "underlined" multiples (in general, depending on variables (x^i, y^4)),

$$\begin{aligned} g_i &= g_i(x^k), g_a = \omega^2(x^i, y^c) h_a(x^k, y^3) \underline{h}_a(x^k, y^4), \\ N_i^3 &= w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4). \end{aligned} \quad (6)$$

Such functions of necessary smooth class have to be defined in a form generating solutions of Einstein eqs. A conformal v -factor $\omega(x^i, y^c)$ may depend on all coordinates. We may simplify substantially the constructions if we take $\omega = \underline{h}_a = 1$ and $\underline{w}_i = \underline{n}_i = 0$ resulting in generic off-diagonal metrics with Killing symmetry on $\partial/\partial y^4$. There will be used brief denotations for partial derivatives: $a^\bullet = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial y^3$, $a^\circ = \partial a/\partial y^4$.³

The system of Einstein eqs derived for the first action in (1), where the Ricci tensor $\widehat{R}_{\alpha\beta}$ is computed for (6), can be transformed equivalently into a system of PDE with

²Such a conventional decomposition can be performed additionally/alternatively / in parallel to the well known 3 + 1 splitting and Arnowit-Deser-Misner, ADM, formalism. Parametrizations of type (3) are used, for instance, in Kaluza-Klein gravity (see review [11]). In this paper, we consider 4-d models with nonlinear dependencies of $N_i^a(u^\beta)$ on all coordinates and do not set compactification on some "extra" dimensions.

³In this work, we restrict our constructions to some classes of spacetime metrics, which can be generated from a set of nontrivial data $[g_i, h_a, w_i, n_i]$ with Killing symmetry by nonholonomic deformations depending on certain small parameters resulting into general "non-Killing" data $[g_i, \omega^2 h_a \underline{h}_a, w_i + \underline{w}_i, n_i + \underline{n}_i]$. In a perturbative approach to QG, we can consider that the contributions from quadratic terms for products of the coefficients of metrics and/or connections, of type $\Gamma \cdot \Gamma$, are small for a fixed open region $U \subset \mathbf{V}$ endowed with normal coordinates.

h-v-decoupling, see details in [8, 9],

$$g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1} = 2g_1 g_2 \hat{\Lambda},$$

$$\frac{-1}{2h_3 h_4} [h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] +$$

$$\frac{1}{2h_3 h_4} [\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2h_3} - \frac{\underline{h}_3^{\circ} \underline{h}_4^{\circ}}{2h_4}] = -\hat{\Lambda}, \quad (7)$$

$$\frac{w_k}{2h_4} [h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4} \left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4} \right)$$

$$- \frac{\partial_k h_4^*}{2h_4} + \frac{h_3}{2h_4} \underline{n}_k^{\circ\circ} + \left(\frac{h_3}{h_4} \underline{h}_4^{\circ} - \frac{3}{2} \underline{h}_3^{\circ} \right) \frac{\underline{n}_k^{\circ}}{2h_4} = 0, \quad (8)$$

$$\frac{\underline{w}_k}{2h_3} [\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^{\circ})^2}{2h_3} - \frac{\underline{h}_3^{\circ} \underline{h}_4^{\circ}}{2h_4}] + \frac{\underline{h}_3^{\circ}}{4h_3} \left(\frac{\partial_k \underline{h}_3}{h_3} + \frac{\partial_k \underline{h}_4}{h_4} \right)$$

$$- \frac{\partial_k \underline{h}_3^{\circ}}{2h_3} + \frac{h_4}{2h_3} \underline{n}_k^{**} + \left(\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^* \right) \frac{\underline{n}_k^*}{2h_3} = 0, \quad (9)$$

$$w_i^* = (\partial_i - w_i) \ln |h_4|, (\partial_k - w_k) w_i = (\partial_i - w_i) w_k, \quad (10)$$

$$n_i^* = 0, \partial_i n_k = \partial_k n_i,$$

$$\underline{w}_i^{\circ} = 0, \partial_i \underline{w}_k = \partial_k \underline{w}_i, \underline{n}_i^{\circ} = (\partial_i - \underline{n}_i) \ln |\underline{h}_3|,$$

$$(\partial_k - \underline{n}_k) \underline{n}_i = (\partial_i - \underline{n}_i) \underline{n}_k,$$

$$\mathbf{e}_k \omega = \partial_k \omega - (w_i + \underline{w}_i) \omega^* - (n_i + \underline{n}_i) \omega^{\circ} = 0. \quad (11)$$

The values \hat{R}_{α}° and \hat{R}_{ak} are equal to the corresponding ones computed for the Levi-Civita connection if the conditions (10) are satisfied. In order to exclude certain "degenerate" classes of solutions of above nonlinear systems of PDEs, we can impose the conditions $h_4^* \neq 0$ and $\underline{h}_3^{\circ} \neq 0$. Such conditions can be satisfied always if corresponding frame/coordinate transforms to necessary ansatz are considered.

At the next step, we show that we can construct in general form a class of exact solutions with generic off-diagonal metrics determined by coefficients with one Killing symmetry, $[g_i, h_a, w_i, n_i]$ with $h_4^* \neq 0$. For

$$\phi(x^k, y^3) = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \quad \alpha_i = h_4^* \partial_i \phi, \quad \beta = h_4^* \phi^*, \quad (12)$$

we can write respectively the eqs (7), (8) in the forms,

$$\phi^* h_4^* = 2h_3 h_4 \hat{\Lambda}, \quad \phi^* \neq 0, \quad (13)$$

$$\beta w_i + \alpha_i = 0. \quad (14)$$

For (9), we must take any trivial solution given by a function $n_i = n_i(x^k)$ satisfying the conditions $\partial_i n_j = \partial_j n_i$ in order to solve the constraints (10). Using coefficients (12) with $\alpha_i \neq 0$ and $\beta \neq 0$, the solution of the above system of Einstein eqs with arbitrary off-diagonal coefficients for one-Killing symmetry can be expressed in the form determined by generating functions $\psi(x^k)$, $\phi(x^k, y^3)$, $\phi^* \neq 0$, $n_i(x^k)$ and $\underline{h}_4(x^k, y^4)$, and integration function ${}^0\phi(x^k)$ following recurrent formulas and conditions,

$$h_3 = \pm \frac{(\phi^*)^2}{4\hat{\Lambda}}, \quad h_4 = \mp \frac{1}{4\hat{\Lambda}} e^{2[\phi - {}^0\phi]}, \quad w_i = -\frac{\partial_i \phi}{\phi^*}. \quad (15)$$

In the above formulas, we should take respective values $\epsilon_i = \pm 1$ and \pm in (15) in order to fix a necessary space-time signature. The generating/ integration functions

may depend also on arbitrary finite sets of parameters $\theta = ({}^1\theta, {}^2\theta, \dots)$ at we found in Ref. [7, 8]. In general, such parametric functions are of type $\psi(x^k, \theta)$, $\phi(x^k, y^3, \theta)$ etc and their explicit form have to be defined from certain boundary/asymptotic conditions and/or experimental data. In some limits, the resulting solutions may describe configurations with effective broken Lorentz symmetry, anisotropies, deformed symmetries etc. For simplicity, we shall not write in explicit form the parametric dependence of values if that will not result in ambiguities. Re-scaling the generating function ϕ in such a form that $\phi^* \rightarrow \phi^* \hat{\Lambda}$ for $\hat{\Lambda} \neq 0$, we can generate in a similar form solutions with Killing symmetry on $\partial/\partial y^3$ with data $[g_i, \underline{h}_a, \underline{w}_i, \underline{n}_i]$ for "rescaled" generating/integration functions $\underline{\phi}(x^k, y^4)$, $\underline{\phi}^{\circ} \neq 0$, $\underline{w}_i(x^k)$, ${}^0\underline{\phi}(x^k)$, when $\underline{h}_3 = \mp e^{2[\underline{\phi} - {}^0\underline{\phi}]}$, $\underline{h}_4 = \pm (\underline{\phi}^{\circ})^2$, $\underline{n}_i = -\partial_i \underline{\phi} / \underline{\phi}^{\circ}$.

We conclude that the Einstein eqs $\hat{R}_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} \hat{\Lambda}$, where \hat{R}_{β}^{α} is the Ricci tensor and $\hat{\Lambda} \neq 0$, decouple in very general forms with respect to N-adapted frames (4) and (5), which allows us to express their solutions (up to corresponding frame transforms) in a form (6),

$$\mathbf{g} = \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \omega^2 [\epsilon_3 (\phi^*)^2 e^{2(\phi - {}^0\phi)} \mathbf{e}^3 \otimes \mathbf{e}^3 +$$

$$\epsilon_4 (\underline{\phi}^{\circ})^2 e^{2(\underline{\phi} - {}^0\underline{\phi})} \mathbf{e}^4 \otimes \mathbf{e}^4], \quad (16)$$

$$\mathbf{e}^3 = dy^3 + (\underline{w}_i - \partial_i \phi / \phi^*) dx^i, \quad \mathbf{e}^4 = dy^4 + (n_i - \partial_i \underline{\phi} / \underline{\phi}^{\circ}) dx^i,$$

where $\epsilon_{\beta} = \pm 1$ fix a corresponding signature and ω is subjected to constraints of type (11),

$$\mathbf{e}_k \omega = \partial_k \omega + (\partial_i \phi / \phi^* - \underline{w}_i) \omega^* + (-n_i + \partial_i \underline{\phi} / \underline{\phi}^{\circ}) \omega^{\circ} = 0.$$

Such classes of metrics are generic off-diagonal and defined by corresponding sets of generating/ integration functions and parameters. Other classes of solutions with $h_4^* = 0$, or $\underline{h}_3^{\circ} = 0$, and/or $\hat{\Lambda} = 0$ consist some special cases analyzed in [8, 9]. Physically important solutions for black holes/ellipsoids, with singularities and horizons, can be constructed for corresponding classes of coefficients of metrics, generating and integration functions etc.

Solutions mimicking non-standard perfect fluid coupling in flat backgrounds.

— The scalar curvature ${}^s\hat{R} := \mathbf{g}^{\alpha\beta} \hat{R}_{4k}$ derived from the Ricci tensors computed for ansatz \mathbf{g} (6) with Killing symmetry on $\partial/\partial y^4$ and nontrivial $[g_i, h_a, w_i, n_i]$ is computed⁴ ${}^s\hat{R} = -2\hat{\Lambda} - \frac{\phi^* h_4^*}{h_3 h_4} = -4\hat{\Lambda}$. Let us state the conditions for generating functions when a metric $\mathbf{g}_{\alpha\beta}$ is defined as a solution of field eqs in theories for ${}^{[1]}\mathcal{L}$ and/or ${}^{[2]}\mathcal{L}$ is also equivalent to a solution derived for ${}^{[3]}\mathcal{L}$. Considering that \tilde{L} is a matter Lagrangian density depending only on the metric tensor components $\mathbf{g}_{\alpha\beta}$ but not on its derivatives, for the energy-momentum tensor $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{|\mathbf{g}|}} \frac{\delta(\sqrt{|\mathbf{g}|} \tilde{L})}{\delta \mathbf{g}^{\mu\nu}} = \mathbf{g}_{\mu\nu} \tilde{L} - 2 \frac{\partial \tilde{L}}{\partial \mathbf{g}^{\mu\nu}}$, we can

⁴for simplicity, hereafter, we shall use such classes of off-diagonal solutions even technically it will be always possible to extend the constructions to general ones (16)

construct sources with nontrivial⁵ $T_1^1 = T_2^2 = \Upsilon(x^i, y^3)$ and $T_3^3 = T_4^4 = \hat{\Lambda}$. For the field eqs derived from $^{[2]}\mathcal{L} = {}^s\tilde{R} + \tilde{L}$, for the same \mathbf{g} and ${}^s\tilde{R} = {}^s\hat{R}$, with respect to N-adapted frames, the first eq from (12) and (13) modify respectively as

$$\sqrt{|h_3 h_4|} = h_4^* e^{-\tilde{\phi}} \text{ and } \tilde{\phi}^* h_4^* = 2h_3 h_4 \Upsilon \quad (17)$$

for a new generating function $\tilde{\phi}(x^i, y^3)$ and $\Upsilon(x^i, y^3)$.

The theories for $^{[1]}\mathcal{L}$ and $^{[2]}\mathcal{L}$ are equivalent if their generating functions and sources are related as $|\Upsilon|^{-1} (e^{2\tilde{\phi}})^* = \hat{\Lambda}^{-1} (e^{2\phi})^*$. Using this formula and solutions (15), we can compute the functional dependencies $h_4[\phi, \hat{\Lambda}] = \pm(4\hat{\Lambda})^{-1} (e^{2\phi} - e^{2\phi})^* = h_4[\tilde{\phi}, \Upsilon]$, $h_3[\tilde{\phi}, \Upsilon] = \pm 4[(\sqrt{|h_4|})^*]^2 e^{-2\tilde{\phi}}$, when $\hat{\Lambda} e^{2\tilde{\phi}} = e^{2\phi} - e^{2\phi} = \hat{\Lambda} \int dy^3 |\Upsilon| (e^{2\phi})^*$ (we can use the inverse formula, $e^{2\phi} - e^{2\phi} = \hat{\Lambda} \int dy^3 |\Upsilon|^{-1} (e^{2\tilde{\phi}})^*$ with ${}^0\tilde{\phi}(x^i)$ and integration functions ${}^0\phi(x^i)$).

At the next step, we state the condition when a source Υ transforms a theory for $^{[2]}\mathcal{L}$ into a model for $^{[3]}\mathcal{L}$. With respect to coordinate frames and for a flat background metric $\eta_{\alpha\beta}$, we consider a generic off-diagonal metric ${}^\diamond\mathbf{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}(x^i, t)$, where $y^3 = t$ is the timelike coordinate, with chosen "gauge" conditions $h_{tt} = h_{t\hat{i}} = h_{\hat{i}t} = 0$, for $\hat{i}, \hat{j} = 1, 2, 4$ (on a manifold \mathbf{V} , we can consider a "double" splitting $(3+1)$ and $(2+2)$). The corresponding Ricci tensor and scalar curvature are

$$\begin{aligned} {}^\diamond R_{\hat{i}\hat{j}} &= \frac{1}{2} \delta_{\hat{i}\hat{j}}^{**} + \partial_{\hat{i}} \partial^{\hat{k}} h_{\hat{j}\hat{k}} + \partial_{\hat{j}} \partial^{\hat{k}} h_{\hat{i}\hat{k}} - \partial_{\hat{k}} \partial^{\hat{k}} h_{\hat{i}\hat{j}}, \\ {}^\diamond R_{33} &= -\frac{1}{2} \delta^{\hat{i}\hat{j}} h_{\hat{i}\hat{j}}^{**}; \quad {}^\diamond R = \delta^{\hat{i}\hat{j}} (h_{\hat{i}\hat{j}}^{**} - \partial_{\hat{k}} \partial^{\hat{k}} h_{\hat{i}\hat{j}}) + \partial^{\hat{i}} \partial^{\hat{i}} h_{\hat{i}\hat{i}}. \end{aligned}$$

We chose a generating function $\tilde{\phi}(x^i, t)$, $\tilde{\phi}^* \neq 0$, when $^{[1]}\mathcal{L} = {}^sR - \hat{\Lambda} = -3\hat{\Lambda} - \frac{\phi^* h_4^*}{h_3 h_4} = {}^{[2]}\mathcal{L} = {}^s\tilde{R} + \tilde{L} = {}^{[3]}\mathcal{L} = {}^\diamond R + {}^\diamond L$, with ${}^\diamond L$ taken for ${}^\diamond\mathbf{g}_{\alpha\beta}$ and an effective non-standard coupling with a fluid configuration, is induced by ϕ for a generic off-diagonal solution of the system (13) and (14). Einstein manifolds encoding fluid like configurations are generated if $\tilde{\phi}$ contains parameters $\alpha, \beta, \rho, \varpi$ introduced into eqs

$$\begin{aligned} \frac{\tilde{\phi}^* h_4^*}{2h_3 h_4} &= -{}^\diamond L = \alpha \rho^2 \{ [\beta(3\varpi - 1) + \frac{\varpi - 1}{2}] \delta^{\hat{i}\hat{j}} h_{\hat{i}\hat{j}}^{**} + \\ &(\varpi + 3\varpi\beta - \beta)(\partial^{\hat{i}} \partial^{\hat{j}} h_{\hat{i}\hat{j}} - \partial_{\hat{k}} \partial^{\hat{k}} h_{\hat{i}\hat{j}}) \}^2. \quad (18) \end{aligned}$$

In above formulas, we wrote ${}^\diamond\Upsilon = -{}^\diamond L$ in order to emphasize that such a source is determined for a metric ${}^\diamond\mathbf{g}_{\alpha\beta}$ and Lagrangian ${}^\diamond L$. Using this expression and formulas (15) redefined for (17), we find recurrently

$$h_4 = \pm \frac{1}{4} \int dy^3 |\Upsilon|^{-1} (e^{2\tilde{\phi}})^*, \quad h_3 = \pm 4 \left[\left(\sqrt{|h_4|} \right)^* \right]^2 e^{-2\tilde{\phi}},$$

⁵we may consider $\tilde{L} = (g_1^{-1} + g_2^{-1}) P_1 + (g_3^{-1} + g_4^{-1}) P_2$ and $(g_1^{-1} + g_2^{-1} - 2) P_1 + (g_3^{-1} + g_4^{-1}) P_2 = \hat{\Lambda}$, $(g_1^{-1} + g_2^{-1}) P_1 + (g_3^{-1} + g_4^{-1} - 2) P_2 = \Upsilon$.

for $\hat{\Lambda} e^{2\tilde{\phi}} = e^{2\phi} - e^{2\phi} \int dy^3 |\Upsilon| (e^{2\phi})^*$ and $w_i = -\frac{\partial_i \phi}{\phi^*}$ used for a $^{[1]}\mathcal{L}$ -theory.

In the limit $\beta \rightarrow (1 - \varpi)/2(3\varpi - 1)$, we can express ${}^\diamond\Upsilon = \alpha \frac{\rho^2}{4} (\varpi + 1)^2 [(\partial^{\hat{i}} \partial^{\hat{j}} h_{\hat{i}\hat{j}} - \partial_{\hat{k}} \partial^{\hat{k}} h_{\hat{i}\hat{j}})]^2$ and generate a class of Einstein manifolds

$$\begin{aligned} \mathbf{g} &= \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \epsilon_3 \left[\left(\sqrt{\left| \int dy^3 |\Upsilon|^{-1} (e^{2\tilde{\phi}})^* \right|} \right)^2 \right. \\ &\quad \left. e^{-2\tilde{\phi}} \mathbf{e}^3 \otimes \mathbf{e}^3 + \epsilon_4 \int dy^3 |4 {}^\diamond\Upsilon|^{-1} (e^{2\tilde{\phi}})^* \mathbf{e}^4 \otimes \mathbf{e}^4 \right], \quad (19) \\ \mathbf{e}^3 &= dy^3 - (\partial_i \phi / \phi^*) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i dx^i, \end{aligned}$$

Such manifolds are with Killing symmetry on $\partial/\partial y^4$ and broken Lorentz invariance because the source ${}^\diamond\Upsilon$ does not contain the derivative with respect to $\partial/\partial y^3 = \partial_t$. Non-Killing configurations of type (16) can be constructed for sources ${}^\diamond\Upsilon(x^i, y^3) + {}^\diamond\Upsilon(x^i, y^4)$ and nontrivial factors $\omega(x^i, y^a)$, as more general classes of solutions.

The fact that such locally anisotropic spacetimes are not Lorentz invariant is not surprising. We have a similar case for the Schwarzschild - de Sitter solutions which are diffeomorphism invariant and with broken Lorentz symmetry. For the class of solutions (19), in the ultraviolet region where momentum \mathbf{k} is large, the second term for the equivalent theory $^{[3]}\mathcal{L}$ gives the propagator $|\mathbf{k}|^{-4}$. The longitudinal modes do not propagate being allowed propagation of the transverse one. Such a behavior is similar to that in a theory with non-standard coupling of gravity with perfect fluid when the energy-momentum tensor $T_{\hat{i}\hat{j}} = p \delta_{\hat{i}\hat{j}} = \varpi \rho \delta_{\hat{i}\hat{j}}$ and $T_{33} = \rho$ (if we treat p, ρ and ϖ as standard fluid parameters and eq of state) in the flat background is computed for ${}^\diamond L = -\alpha(T^{\alpha\beta} {}^\diamond R_{\alpha\beta} + \beta T_{\alpha}^{\alpha} {}^\diamond R_{\beta}^{\beta})^2$.

Considering off-diagonal solutions for theories with $^{[1]}\mathcal{L}$ and/or $^{[2]}\mathcal{L}$ derived for generating functions and sources of type (18), we obtain a situation when generic off-diagonal interactions in gravity induce a kind of spontaneous violation of symmetry which is typical in quantum field theories. Using such nonholonomic configurations, we model a vacuum gravitational aether via analogous coupling with non-standard fluid which breaks the Lorentz symmetry for an effective equivalent theory $^{[3]}\mathcal{L}$. This allows us to elaborate a (power-counting) renormalizable model of QG.⁶

Effective renormalizable EYM configurations and modified gravity.

— The constructions with non-standard fluid coupling, effective renormalizability etc performed after (17) were provided, for simplicity, for a flat background. Nevertheless, the approach can be extended to curved backgrounds and generic off-diagonal gravitational-field interactions. Using the principle of relativity, we can work in a local Lorenz frame when, for instance, the effective fluid does not flow. This way we preserve unitarity and axioms of GR working with general

⁶In this letter, we do not prove explicitly the renormalizability of off-diagonal solutions but show that they can be associated/ related to certain "renormalizable" quantum models studied by other authors.

class of solutions for theories $^{[1]}\mathcal{L}$ and/or $^{[2]}\mathcal{L}$. Via respective parametric dependence and nonholonomic transforms we mimic some models $^{[3]}\mathcal{L}$ with anisotropic coupling. More than that, even the value ${}^\diamond\Upsilon$ can be fixed to not contain time derivatives, it will get such terms in arbitrary frames of reference. As we mentioned in [8, 9], the solutions of type (19) and (16) in the diagonal spherical symmetry limit (here we include the condition $T^{\alpha\beta} = 0$) contain the Schwarzschild and Kerr black hole/ellipsoid metrics. The analyzed models with $\Upsilon = {}^\diamond\Upsilon = -{}^\diamond L$ (18) result in $z = 2$ Hořava–Lifshitz theories.

Let us show how in the scheme of Lagrangian densities (1) we can include $z = 3$ theories which allows us to generate ultra-violet power counting renormalizable 3+1 and/or 2+2 quantum models. Instead of ${}^\diamond L$, we can take more general sources and generating functions, $-{}^\diamond_n L =$

$$\alpha\{(T^{\mu\nu} {}^\diamond\nabla_\mu {}^\diamond\nabla_\nu + \gamma T^\alpha_\alpha {}^\diamond\nabla^\beta {}^\diamond\nabla_\beta)^n (T^{\alpha\beta} {}^\diamond R_{\alpha\beta} + \beta T^\alpha_\alpha {}^\diamond R^\beta_\beta)\}^2, \quad (20)$$

where n and γ are constants. Introducing $\Upsilon = -{}^\diamond_n L$ in formulas (18) in order to state a different class of generating functions ${}^n\tilde{\phi}$ from $\frac{{}^n\tilde{\phi}^* h^*_4}{2h_3 h_4} = -{}^\diamond_n L$ and construct off-diagonal solutions of type (19) for data $\tilde{\phi} \rightarrow {}^n\tilde{\phi}, \phi \rightarrow {}^n\phi$ and ${}^\diamond\Upsilon \rightarrow -{}^\diamond_n L$. In general, we can consider noninteger values for n , for instance, $n = 1/2, 3/2$ etc. If n is negative, we can generate non-local theories determined by a respective off-diagonal polarization of vacuum and cosmological constant in GR (the question of "renormalizability" is more complex for such cases). Following analysis provided in [7] the analogous $^{[3]}\mathcal{L}$ and $^{[4]}\mathcal{L}$ models derived for (20) are renormalizable if $n = 1$ and super-renormalizable for $n = 2$. The values induced by a nontrivial source/Lagrangian density ${}^\diamond_n L$ contains higher derivative terms and effectively break the Lorenz symmetry for high energies, i.e. in UV region. In the IR, we positively get the usual Einstein gravity. Nevertheless, in our approach, GR is not only a limit from certain modifications with non-standard coupling of type (18) and/or (20) but a model with possible "branches" of complexity, anisotropies, inhomogeneities and Lorentz violations depending on parameters and generating functions. Certain families of solutions are (super) renormalizable because of off-diagonal nonlinear interactions of gravitational and effective matter fields.

It is important to analyze two special cases for sources ${}^\diamond_n L$, for instance, in the fluid approximation for matter. We model two theories $^{[1]}\mathcal{L}$ and $^{[3]}\mathcal{L}$ with cosmological constant when both $\varpi = -1$ and $\hat{\Lambda} \neq 0$. This corresponds to diagonal solutions with $T^{\alpha\beta} {}^\diamond R_{\alpha\beta} + \beta T^\alpha_\alpha {}^\diamond R^\beta_\beta = 0$ when $1/|k|^4$ can not be obtained for the propagator. To correct the model, we have to introduce additional off-diagonal terms. The case $\varpi = 1/3$ corresponds to the radiation of conformal matter with divergent β . This also does not provide "good" solutions if other "anisotropic" contributions are not considered.

The approach with generic off-diagonal solutions modelling effective covariant renormalizable theories can be ex-

tended for gravitational interactions with a non-Abelian SU(2) gauge field $\mathbf{A} = \mathbf{A}_\mu \mathbf{e}^\mu$ coupled to a triplet Higgs field Φ , see details in [8, 9]. In order to prove the decoupling property, we have to use a covariant operator $\hat{\mathbf{D}}$ which is adapted to the N-splitting. It is not possible to "see" a general splitting of Einstein and matter field eqs if we do not consider 2 + 2 spacetime decompositions with such non-integrable distributions. The linear connection $\hat{\mathbf{D}}$ is equivalent to the Levi-Civita connection ∇ if the conditions (10) are satisfied and all computations are performed with respect N-adapted bases. In terms of $\hat{\mathbf{D}}$, such nonholonomic interactions are described

$$\hat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} {}^s\hat{R} = 8\pi G ({}^H T_{\beta\delta} + {}^Y M T_{\beta\delta}), \quad (21)$$

$$D_\mu(\sqrt{|g|}F^{\mu\nu}) = \frac{1}{2}ie(\sqrt{|g|})[\Phi, D^\nu\Phi], \quad (22)$$

$$D_\mu(\sqrt{|g|}\Phi) = \lambda(\sqrt{|g|})(\Phi^2_{[0]} - \Phi^2)\Phi, \quad (23)$$

where the source of the Einstein eqs is

$${}^H T_{\beta\delta} = Tr[\frac{1}{4}(\nabla_\delta\Phi \nabla_\beta\Phi + \nabla_\beta\Phi \nabla_\delta\Phi) - \frac{1}{4}\mathbf{g}_{\beta\delta}\nabla_\alpha\Phi \nabla^\alpha\Phi] \quad (24)$$

$$-\mathbf{g}_{\beta\delta}\mathcal{V}(\Phi), \quad {}^Y M T_{\beta\delta} = 2Tr(\mathbf{g}^{\mu\nu}F_{\beta\mu}F_{\delta\nu} - \frac{1}{4}\mathbf{g}_{\beta\delta}F_{\mu\nu}F^{\mu\nu}). \quad (25)$$

The curvature of gauge field \mathbf{A}_δ is $F_{\beta\mu} = \mathbf{e}_\beta\mathbf{A}_\mu - \mathbf{e}_\mu\mathbf{A}_\beta + ie[\mathbf{A}_\beta, \mathbf{A}_\mu]$, where e is the coupling constant, $i^2 = -1$, and $[\cdot, \cdot]$ is used for the commutator. $\Phi_{[0]}$ in (23) is the vacuum expectation of the Higgs field which determines the mass ${}^H M = \sqrt{\lambda}\eta$, when λ is the constant of scalar field self-interaction with potential $\mathcal{V}(\Phi) = \frac{1}{4}\lambda Tr(\Phi^2_{[0]} - \Phi^2)^2$. The gravitational constant G defines the Planck mass $M_{Pl} = 1/\sqrt{G}$; the is a nontrivial mass of gauge boson, ${}^W M = ev$.

A class of diagonal solutions of the system (21)–(23), see Ref. [12], is given by metric ansatz ${}^\diamond\mathbf{g} =$

$${}^\diamond g_i(x^1)dx^i \otimes dx^i + {}^\diamond h_a(x^1, x^2)dy^a \otimes dy^a = q^{-1}(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\varphi \otimes d\varphi - \sigma^2(r)q(r)dt \otimes dt, \quad (26)$$

where the coordinates and metric coefficients are parametrized, respectively, $u^\alpha = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$ and ${}^\diamond g_1 = q^{-1}(r)$, ${}^\diamond g_2 = r^2$, ${}^\diamond h_3 = r^2 \sin^2 \theta$, ${}^\diamond h_4 = -\sigma^2(r)q(r)$, for $q(r) = 1 - 2m(r)/r - \Lambda r^2/3$, where Λ is a cosmological constant. The function $m(r)$ is usually interpreted as the total mass-energy within the radius r which for $m(r) = 0$ defines an empty de Sitter, dS , space written in a static coordinate system with a cosmological horizon at $r = r_c = \sqrt{3/\Lambda}$. An ansatz for solution of Yang–Mills (YM) eqs on (21) on a spherically symmetric background in GR is defined by a single magnetic potential $v(r)$, ${}^\diamond A = {}^\diamond A_2 dx^2 + {}^\diamond A_3 dy^3 = \frac{1}{2e}[v(r)\tau_1 d\theta + (\cos \theta \tau_3 + v(r)\tau_2 \sin \theta) d\varphi]$, where τ_1, τ_2, τ_3 are Pauli matrices, and for eqs (23) of the Higgs (H) field is given by $\Phi = {}^\diamond\Phi = \chi(r)\tau_3$. The functions $\sigma(r), q(r), v(r), \chi(r)$ can be computed for data $[{}^\diamond\mathbf{g}(r), {}^\diamond A(r), {}^\diamond\Phi(r)]$. For instance, the diagonal Schwarzschild–de Sitter solution of (21)–(23) is that determined by data $v(r) = \pm 1, \sigma(r) = 1, \phi(r) = 0, q(r) =$

$1 - 2M/r - \Lambda r^2/3$ defining a black hole configuration inside a cosmological horizon because $q(r) = 0$ has two positive solutions and $M < 1/3\sqrt{\Lambda}$.

A "prime" diagonal solution ${}^\circ\mathbf{g}$ (26) can be transformed into "target" off-diagonal metrics $\mathbf{g} = {}^\eta\mathbf{g}$, ${}^\circ\mathbf{g} \rightarrow {}^\eta\mathbf{g}$,

$$\begin{aligned} {}^\eta\mathbf{g} &= \eta_i e^\psi dx^i \otimes dx^i + \omega^2 [\eta_3 (\phi^*)^2 e^{2(\phi - {}^\circ\phi)} \mathbf{e}^3 \otimes \mathbf{e}^3 + \\ &\quad \eta_4 (\phi^*)^2 e^{2(\phi - {}^\circ\phi)} \mathbf{e}^4 \otimes \mathbf{e}^4], \\ \mathbf{e}^3 &= dy^3 + \eta_i^3 (w_i + \underline{w}_i) dx^i, \quad \mathbf{e}^4 = dy^4 + \eta_j^4 (n_j + \underline{n}_j) dx^j, \end{aligned} \quad (27)$$

where the gravitational η -polarizations $\eta_i(x^k)$, $\eta_b(x^k, y^a)$ and $\eta_j^c(x^k, y^a)$ have to be found from the condition that such metrics generate solutions of (21)–(23). With respect to N-adapted frames, the gauge fields are deformed

$$A_\mu(x^i, y^3) = {}^\circ A_\mu(x^1) + {}^\eta A_\mu(x^i, y^a), \quad (28)$$

where ${}^\eta A_\mu$ is a function for which $F_{\beta\mu} = {}^\circ F_{\beta\mu}(x^1) + {}^\eta F_{\beta\mu}(x^i, y^a) = s\sqrt{|g|}\varepsilon_{\beta\mu}$, for $s = \text{const}$ and $\varepsilon_{\beta\mu}$ being the absolute antisymmetric tensor. Such a tensor always solve the eqs $D_\mu(\sqrt{|g|}F^{\mu\nu}) = 0$, which always give us the possibility to determine the distortions ${}^\eta F_{\beta\mu}$, and ${}^\eta A_\mu$, for any given ${}^\circ A_\mu$ and/or ${}^\circ F_{\beta\mu}$. The scalar field is nonholonomically modified ${}^\circ\Phi(x^1) \rightarrow \Phi(x^i, y^a) = {}^\Phi\eta(x^i, y^a) {}^\circ\Phi(x^1)$ by ${}^\Phi\eta$ is such a way that $D_\mu\Phi = 0$ and $\Phi(x^i, y^a) = \pm\Phi_{[0]}$. This nonholonomic configuration of the nonlinear scalar field is not trivial even with respect to N-adapted frames $\mathcal{V}(\Phi) = 0$ and ${}^H T_{\beta\delta} = 0$, see (24).⁷

The gauge fields with potential A_μ (28) modified nonholonomically by Φ determine exact solutions for the YMH system (22) and (23) for spacetime metrics type (27). The corresponding stress energy–momentum tensor is computed (see details in sections 3.2 and 6.51 from [13]) ${}^{YM}T_\beta^\alpha = -4s^2\delta_\beta^\alpha$, which means that nonholonomically interacting gauge and Higgs fields, with respect to N-adapted frames, result in an effective cosmological constant ${}^s\lambda = 8\pi s^2$ which should be added to a respective sources of Einstein eqs. Using above presented constructions, we conclude that various classes of nonholonomic EYM systems can be modeled, respectively, as theories of type ${}^{[1]}\mathcal{L}$, ${}^{[2]}\mathcal{L}$ and ${}^{[3]}\mathcal{L}$ using an effective cosmological constant $\hat{\Lambda} = \Lambda + {}^s\lambda$ and off-diagonal ansatz for metrics (27) when η -polarizations are treated as additional generating functions. Instead of such classes of equivalence of conjectured covariantly renormalizable theories with locally anisotropic interactions, we can reformulate the results in the language of $F(R)$ -gravity of Hořava–Lifshitz type [10]. The effective Lagrangian density ${}^{[4]}\mathcal{L} = F(\hat{R})$ can be chosen for $\hat{R} = R + {}^\circ L$, see (18), when $z = 2$, or $\hat{R} = R + {}^\circ_n L$, see (20), when $z = 2n + 2$. EYM configurations are expected to be covariantly renormalizable when $z \geq 3$ which may be demonstrated using arguments presented in [7, 14] and in this paper and/or other variants of

anisotropic renormalization. A rigorous proof is possible if we chose generating functions when our off-diagonal metrics induce models with covariant power-counting renormalization [15–17]; such constructions will be provided in our future works.

Concluding Remarks. – In summary, we have used the possibility to decouple in very general forms the gravitational field eqs in GR and construct various classes of exact generic off-diagonal solutions in order to elaborate covariant (super) renormalizable theories of gravity. We also briefly considered how EYM interactions can be encoded into nonholonomic Einstein manifolds and effective models of interactions with broken Lorenz symmetry and covariant renormalization.

We emphasize that in this work we followed an "orthodox" approach to modeling classical and quantum geometries and physical theories of interactions keeping the constructions and physical paradigm to be maximally closed to Einstein gravity. Our opinion is that the bulk of experimental data for modern gravity and cosmology can be explained/predicted using generic off-diagonal solutions in GR and their quantized versions.

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⁷For the ansatz (27), the Higgs eqs are $(\partial/\partial x^i - A_i)\Phi = (w_i + \underline{w}_i)\Phi^* + (n_i + \underline{n}_i)\Phi^0$, $(\partial/\partial y^3 - A_3)\Phi = 0$, $(\partial/\partial y^4 - A_4)\Phi = 0$, i. e. a scalar field Φ modifies the off-diagonal components of the metric via $w_i + \underline{w}_i$ and $n_i + \underline{n}_i$ and nonholonomic conditions for $A_\mu = {}^\eta A_\mu$.